Announcements

1) In Webwork,

"Singular" means

"not invertible".

2) Colloquium, 3-4, CB 2046

On mathematical physics

uses linear algebra!

Notation: (inverses)

If A is an $n \times n$ matrix and A is invertible with respect to multiplication (or just "invertible"), we write A^{-1} for the inverse.

More Notation $M_n(\mathbb{R})$ means all $n \times n$ real-valued matrices.

Definition: (permutation)

Let $T_n = \{1, 2, 3, \ldots, n\}$,

i.e. T_n is a set with n elements in it, indexed by $1, 2, 3, \ldots, n$.

A permutation is a function $\varphi : T_n \to T_n$ that is

one-to-one: if $\varphi(i) = \varphi(j)$, then $i = j$.

Example 1

$$T_2 = \{1, 2\}$$

There are two permutations of T_2, φ and ψ

- $\varphi(1) = 1$ $\quad$ $\varphi(2) = 2$
- $\psi(1) = 2$ $\quad$ $\psi(2) = 1$

$T_3 = \{1,2,3\}$.

There are $6 = 3!$ different permutations. Here are some:

- $\varphi(1)=1$, $\varphi(2)=2$, $\varphi(3)=3$

- $\psi(1)=2$, $\psi(2)=3$, $\psi(3)=1$

- $\gamma(1)=2$, $\gamma(2)=1$, $\gamma(3)=3$

In general, there are $n!$ permutations on I_n.

Definition: (flips)

A **flip** on T_n is a permutation that only changes 2 elements.

Example 2

In T_2,

$\psi(1)=2$, $\psi(2)=1$

is a flip.

In T_3,

$\gamma(1)=2$, $\gamma(2)=1$, $\gamma(3)=3$

is a flip, but

$\psi(1)=2$, $\psi(2)=3$, $\psi(3)=1$

is not a flip.

In T_5,

$\varphi(1)=5$, $\varphi(2)=2$, $\varphi(3)=3$,
$\varphi(4)=4$, $\varphi(5)=1$

is a flip, but

$\psi(1)=5$, $\psi(2)=4$, $\psi(3)=2$,
$\psi(4)=1$, $\psi(5)=3$

is not.

Definition: (sign of a permutation)

If φ is a permutation, define the sign of φ to be 1 if φ is composed of an even number of flips and -1 if φ is composed of an odd number of flips. Write $\text{sign}(\varphi)$ for the sign.

Note: (multiplicativity) If φ, ψ are two permutations of $\overline{I_n}$,

$$\text{sign}(\varphi \circ \psi) = \text{sign}(\varphi) \cdot \text{sign}(\psi)$$

Notation: If φ is a flip on T_n that flips the numbers k and m, write $\varphi = (km)$.

Example 3:

φ on T_2

$\varphi(1)=1 \quad \varphi(2)=2.$

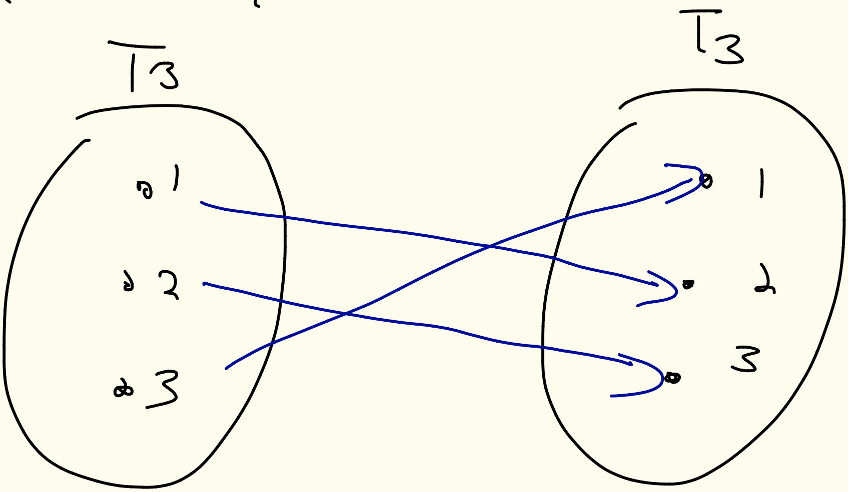
$$\varphi = (12)\circ(12)$$

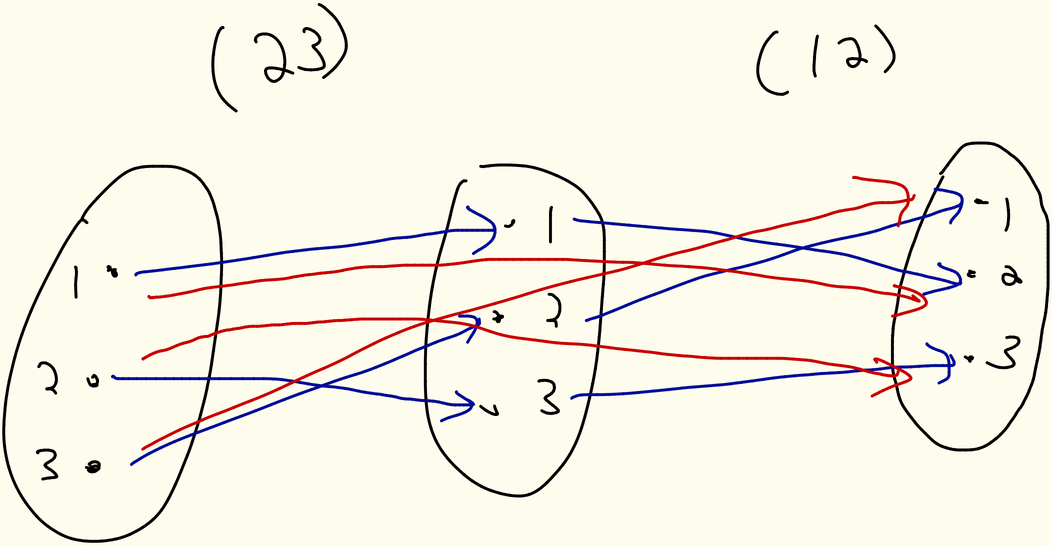
This says $\mathrm{sign}(\varphi)=1$.

ψ on T_3

$\psi(1)=2, \psi(2)=3, \psi(3)=1$

Claim: $\psi=(12)\circ(23)$





Same picture as ψ,

so $\psi = (12) \circ (23)$

On T_5

$\omega(1) = 3$, $\omega(2) = 5$,
$\omega(3) = 4$, $\omega(4) = 2$, $\omega(5) = 1$

Show that

$$\omega = (13) \circ (34) \circ (42) \circ (25) \circ (51)$$

Definition : (determinant)

Let A be in $M_n(\mathbb{R})$.

Write $A = (a_{i,j})_{i,j=1}^{n}$

($a_{i,j}$'s are entries of A)

define

$$\det(A) = \sum_{\substack{\sigma \text{ permutation} \\ \text{of } T_n}} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

the determinant of A.

Example 4:

Let A be in $M_2(\mathbb{R})$,

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

There are two permutations of T_2, $\underbrace{(12)}_{\sigma}$ and $\underbrace{(12)\circ(12)}_{\psi} = \text{identity}$.

Then

$$\det(A) = \operatorname{sign}(\sigma) a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} + \operatorname{sign}(\psi) a_{1,\psi(1)} a_{2,\psi(2)}$$

$$\det(A) = \operatorname{sign}(\sigma) a_{1,\sigma(1)} \cdot a_{2,\sigma(2)}$$
$$+ \operatorname{sign}(\psi) a_{1,\psi(1)} a_{2,\psi(2)}$$

$$= -1 \cdot a_{1,2} \cdot a_{2,1}$$
$$+ 1 \cdot a_{1,1} a_{2,2}$$

$$= a_{1,1} a_{2,2} - a_{1,2} a_{2,1}.$$

If we write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

the formula becomes

$$\boxed{\det(A) = ad - bc}$$

Theorem: (invertibility)

A matrix B in $M_n(\mathbb{R})$ is invertible precisely when $\det(B) \neq 0$.

Example 5:

Is $A = \begin{bmatrix} -1 & 3 \\ 16 & 22 \end{bmatrix}$

invertible?

$$\det(A) = -22 - 48$$

$$= -70 \neq 0$$

So A is invertible.

Let $A = \begin{bmatrix} -1 & 3 & 6 \\ 0 & 2 & -5 \\ -1 & 7 & -4 \end{bmatrix}$.

Is A invertible?

Calculate $\det(A)$. To do this write

$$\begin{matrix} -1 & 3 & 6 & -1 & 3 \\ 0 & 2 & -5 & 0 & 2 \\ -1 & 7 & -4 & -1 & 7 \end{matrix}$$

$$\begin{array}{ccccc} -1 & 3 & 6 & -1 & 3 \\ 0 & 2 & -5 & 0 & 2 \\ -1 & 7 & -4 & -1 & 7 \end{array}$$

Multiply along diagonals, add the blues, subtract the reds.

$$\det(A) = 8 + 15 + 0 + 12 - 35$$
$$= 0$$

So A is not invertible. Or...

use Wolfram Alpha.