

Announcements

1) In Webwork,

"singular" means

"not invertible".

2) Colloquium, 3-4, CB 2046

On mathematical physics

uses linear algebra!

Notation: (inverses)

If A is an $n \times n$ matrix and A is invertible with respect to multiplication (or just "invertible"), we

write A^{-1} for the inverse.

More Notation $M_n(\mathbb{R})$ means all $n \times n$ real-valued matrices.

Definition: (permutation)

Let $T_n = \{1, 2, 3, \dots, n\}$,

i.e. T_n is a set with n elements in it, indexed by $1, 2, 3, \dots, n$.

A permutation is a function

$\phi : T_n \rightarrow T_n$ that is

one-to-one: if $\phi(i) = \phi(j)$,

then $i = j$.

Example 1

$$T_2 = \{1, 2\}$$

There are two permutations
of T_2 , φ and ψ

— $\varphi(1) = 1 \quad \varphi(2) = 2$

— $\psi(1) = 2 \quad \psi(2) = 1$

$$T_3 = \{1, 2, 3\}.$$

There are $6 = 3!$ different permutations. Here are some:

- $\varphi(1) = 1, \varphi(2) = 2, \varphi(3) = 3$
- $\psi(1) = 2, \psi(2) = 3, \psi(3) = 1$
- $\gamma(1) = 2, \gamma(2) = 1, \gamma(3) = 3$

In general, there are

$n!$ permutations on

$\overline{T_n}$.

Definition: (flips)

A flip on T_n is
a permutation that
only changes 2 elements.

Example 2

In T_2 ,

$$\psi(1)=2, \psi(2)=1$$

is a flip.

In T_3 ,

$$\gamma(1)=2, \gamma(2)=1, \gamma(3)=3$$

is a flip, but

$$\psi(1)=2, \psi(2)=3, \psi(3)=1$$

is **not** a flip.

In T_5 ,

$$\begin{aligned}\varphi(1) &= 5, \quad \varphi(2) = 2, \quad \varphi(3) = 3, \\ \varphi(4) &= 4, \quad \varphi(5) = 1\end{aligned}$$

is a flip, but

$$\begin{aligned}\psi(1) &= 5, \quad \psi(2) = 4, \quad \psi(3) = 2, \\ \psi(4) &= 1, \quad \psi(5) = 3\end{aligned}$$

is not.

Definition: (sign of a permutation)

If φ is a permutation,
define the **sign** of φ to
be 1 if φ is composed
of an even number of flips
and -1 if φ is composed
of an odd number of flips.

Write $\text{sign}(\varphi)$ for the sign.

Note: (multiplicativity) If φ, ψ
are two permutations of $\overline{\mathbb{N}_n}$,
 $\text{sign}(\varphi \circ \psi) = \text{sign}(\varphi) \cdot \text{sign}(\psi)$

Notation: If φ is

a flip on T_n that

flips the numbers k and m ,

write $\varphi = (km)$.

Example 3:

φ on T_2

$$\varphi(1) = 1 \quad \varphi(2) = 2.$$

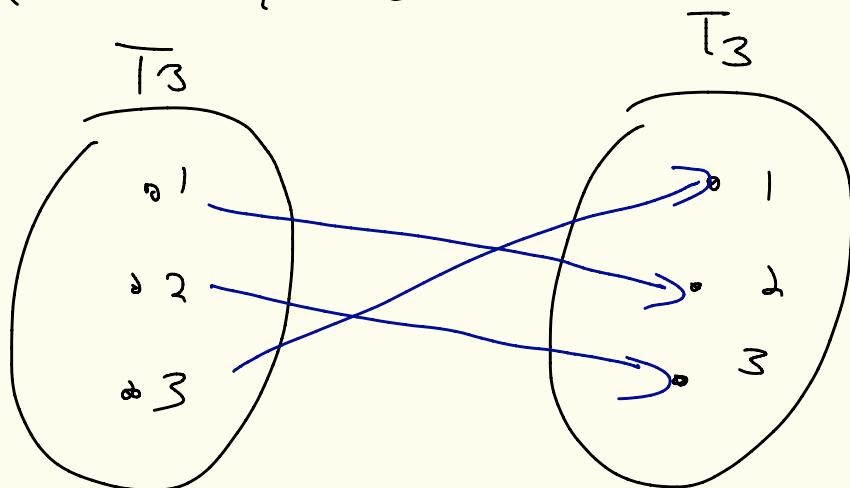
$$\varphi = (12) \circ (12)$$

This says $\text{sign}(\varphi) = 1$.

ψ on T_3

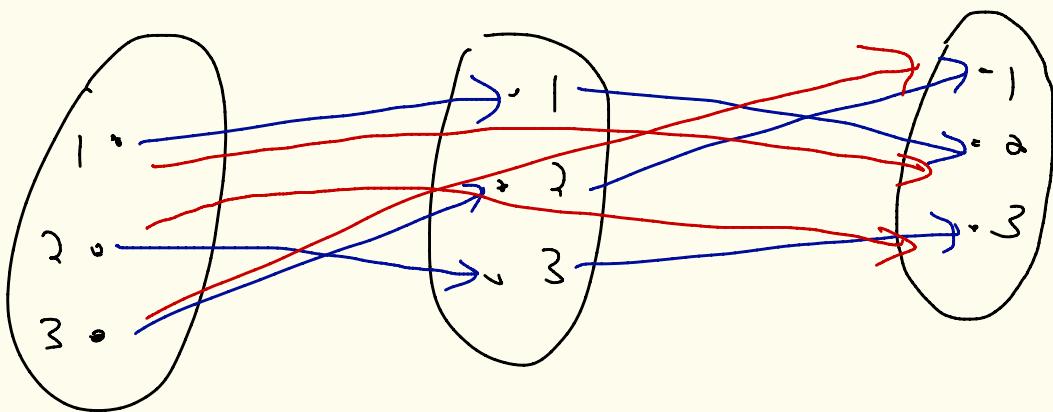
$$\psi(1)=2, \psi(2)=3, \psi(3)=1$$

Claim: $\psi = (12) \circ (23)$



(23)

(12)



Same picture as ψ ,

$$\text{so } \psi = (12) \circ (23)$$

On T_5

$$\omega(1) = 3, \quad \omega(2) = 5,$$

$$\omega(3) = 4, \quad \omega(4) = 2, \quad \omega(5) = 1$$

Show that

$$\omega = (13) \circ (34) \circ (42) \circ (25) \circ (51)$$

Definition : (determinant)

Let A be in $M_n(\mathbb{R})$.

Write $A = (a_{ij})_{i,j=1}^n$

(a_{ij} 's are entries of A)

define

$$\det(A) = \sum_{\sigma \text{ permutation}} \text{Sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

σ permutation
of T_n

the determinant of A .

Example 4:

Let A be in $M_2(\mathbb{R})$,

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

There are two permutations
of T_2 , $\underbrace{(12)}_{\sigma}$ and $\underbrace{(12) \circ (12)}_{\psi} = \text{identity}$.

Then

$$\det(A) = \text{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} + \text{sign}(\psi) a_{1,\psi(1)} a_{2,\psi(2)}$$

$$\det(A) = \text{sign}(\sigma) a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \\ + \text{sign}(\psi) a_{1,\psi(1)} \cdot a_{2,\psi(2)}$$

$$= -1 \cdot a_{1,2} \cdot a_{2,1}$$

$$+ 1 \cdot a_{1,1} a_{2,2}$$

$$= a_{1,1} a_{2,2} - a_{1,2} a_{2,1}$$

If we write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

the formula becomes

$$\boxed{\det(A) = ad - bc}$$

Theorem: (invertibility)

A matrix B in $M_n(\mathbb{R})$

is invertible precisely

when $\det(B) \neq 0$.

Example 5:

Is $A = \begin{bmatrix} -1 & 3 \\ 16 & 22 \end{bmatrix}$

invertible?

$$\begin{aligned}\det(A) &= -22 - 48 \\ &= -70 \neq 0\end{aligned}$$

so A is invertible.

$$\text{Let } A = \begin{bmatrix} -1 & 3 & 6 \\ 0 & 2 & -5 \\ -1 & 7 & -4 \end{bmatrix}.$$

Is A invertible?

Calculate $\det(A)$. To do
this write

The diagram illustrates the expansion of a 3x3 matrix by cofactors. The matrix is:

$$\begin{bmatrix} -1 & 3 & 6 \\ 0 & 2 & -5 \\ -1 & 7 & -4 \end{bmatrix}$$

Red arrows point from the first column to the first row, indicating the expansion along the first column. Blue arrows point from the second column to the second row, indicating the expansion along the second column. The numbers -1, 3, 6, 0, 2, -5, -1, 7, -4 are placed above their respective matrix elements to show they are part of the cofactor calculation.

$$\begin{array}{cccc}
 -1 & 3 & 6 & -1 \\
 0 & 2 & -5 & 2 \\
 -1 & 7 & -4 & -1 \\
 \hline
 \end{array}$$

Diagram illustrating the calculation of the determinant of a 3x3 matrix. The matrix is shown with red and blue lines and arrows indicating the paths for calculating the sum of products of elements along diagonals.

Multiply along diagonals,
add the blues, subtract the
reds.

$$\begin{aligned}
 \det(A) &= 8 + 15 + 0 + 12 - 35 \\
 &= 0
 \end{aligned}$$

So A is not invertible. Or...
use Wolfram Alpha.